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# Deformed Multiconformal Algebra and Its q-Operator Product Expansion

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The q-deformed multiconformal algebra is derived. The $gl_q(n)$-covariant oscillator realization is given in the centerless case. The q-operator product expansion realization is also given.

## 1. INTRODUCTION

In the last decade, a great deal of attention has been paid to the study of quantum groups and quantum algebras (Drinfel'd, 1986; Jimbo, 1985, 1986; Mànin, 1988, 1991, Faddeev *et al.*, 1987; Woronowicz, 1987; Pusz and Woronowicz, 1989; Wess and Zumino, 1990). These new mathematical structures have been applied to physical models, e.g., in conformal field theories (Alvarez-Gaumé *et al.*, 1989; Moore and Reshetikhin, 1989; Bernard and LeClaire, 1991) as well as in the vertex and spin models (De Vega, 1989, Pasquier and Saleurs, 1990) and to quantum topics (see, e.g., Chaichian *et al.*, 1990). The q-deformation of creation and annihilation operators (Macfarlane, 1989; Biedenharn, 1989; Chaichian and Kulish, 1990; Kulish and Damaskinsky, 1990) was applied to construct different q-(super)algebras and generalized statistics with violation of the Pauli principle (Greenberg, 1990; Mohapatra, 1990). Recently, Fleury *et al.* (1995) introduced multiconformal symmetry in order to extend the infinite-conformal symmetry to $n$-dimensional space ($n > 2$). Hence, the scale invariance in such a space can be applied to the theory of ($n - 1$)-branes which are ($n - 1$)-dimensional extended objects generalizing strings; these results can also be applied to critical phenomena

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in more than two dimensions, as well as in the theory of interacting spins more general than Ising's model. The main idea of the present work is to introduce the deformed version of multiconformal algebra and give its realization in term of $q$-operator product expansions of the componants of the q-energy-momentum tensor $T(z^l)$, $l = 0, 1, \ldots, n - 1$.

The organization of this paper is as follows: In Section 2, we introduce the multiconformal transformations and present a model of field theory describing the bosomic case. In Section 3, we present the deformation of centerless multiconformal algebra and its $gl_q(n)$-covariant oscillator realization. Section 4 constructs the q-operator product expansion. Finally, Section 5 gives our conclusion and outlook.

## 2. CLASSICAL MULTICONFORMAL ALGEBRA

In this section, we briefly sketch the multiconformal transformations, also called by Fleury *et al.* (1995) "conformal-like transformations." These are the transformations which leave the $n$-metric tensor invariant up to a scale factor on the multicomplex plane $\mathbf{MC_n}$ (Fleury *et al.*, 1993):

$$\mathbf{MC_n} = \sum_{j=0}^{n-1} x_j e^j, \qquad x_j \in \mathbf{R} \tag{1}$$

where $e$ is the fundamental unit satisfying the basic relation $e^n = -1$. As in the complex plane $\mathbf{C}$, an element $Z \in \mathbf{MC_n}$ possesses an $n$-conjugate

$$z^{(l)} = \sum_{j=0}^{n-1} x_j \omega^{-jl} e^j \tag{2}$$

where $\omega = \exp(2\pi i/n)$.

Let us now introduce the $l$th differential operators, which will be useful in what follows:

$$\partial^{(l)} = \frac{1}{n} \sum_{j=0}^{n-1} \omega^{-jl} e^{-j} \frac{\partial}{\partial x_j} \tag{3}$$

The transformation $z^{(l)} \to F_l(z^{(k)})$ leaves the $n$-metric tensor invariant up to a scale factor if and only if (Fleury *et al.*, 1995)

$$\frac{\partial F_l}{\partial z^{(k)}} = 0, \qquad k \neq l \tag{4}$$

Now, consider the infinitesimal transformations $z^{(l)} \to z'^{(l)} = z^{(l)} + \epsilon f^l(z^{(l)})$ for $n$ even. After a Laurent development of $f^l$ we obtain $n$ copies of the Virasoro algebras without central charge

$$[L_m^{(i)}, L_n^{(j)}] = \delta^{ij} (m - n) L_{m+n}^{(j)} \tag{5}$$

where $i, j = 0, 1, \ldots, n - 1$.

We note that the central charge of such an algebra exists only in the case where $i = j$, and we obtain

$$[L_m^{(i)}, L_n^{(j)}] = \delta^{ij}(m - n) L_{m+n}^{(j)} + \delta^{ij} c^j \frac{m^3 - m}{12} \delta_{m+n,0}^j \tag{6}$$

To end this section, we to note that in our previous paper (Ouarab *et al.*, 1998), we gave the fundamentals of multiconformal quantum field theory; namely the Hermitian action for a multicomplex field $\psi(z^{(0)}, z^{(1)}, \ldots, z^{(n-1)})$ defined on the multicomplex plane. The action $S[\psi]$ reads

$$S[\psi] \sim \int_{MC_n} d^n s\, J^{kl}\, \partial_k\psi\partial_l\psi \tag{7}$$

where $d^n s = dz^{(0)} \wedge dz^{(1)} \wedge \ldots \wedge dz^{(n-1)}$ and $J^{kl}$ is an $n \times n$ matrix such that $J^{kl} =$ antidiagonal $(1/n, 1/n, \ldots, 1/n)$, which obviously reduces to the known Polyakov one for $n = 2$.

## 3. DEFORMED MULTICONFORMAL ALGEBRA AND ITS $gl_q(n)$-COVARIANT OSCILLATOR REALIZATION

To begin with, let us consider the set of q-deformed centerless multi-Virasoro generators which satisfy the relations

$$\begin{aligned} &[L_m^{(i)}, L_n^{(i)}]_{(q^{(n-m)}, q^{(m-n)})} = [n - m]_q\, L_{m+n}^{(i)}, \qquad i = 0, 1, \ldots, n - 1 \\ &L_m^{(i)}\, L_n^{(j)} = q^{nm} L_n^{(j)}\, L_m^{(i)}, \qquad i < j \end{aligned} \tag{8}$$

where $[x]_q = (q^x - q^{-x})/(q - q^{-1})$.

A realization of this $q$-algebra is given with the help of the $gl_q(n)$-covariant oscillator algebra (Pusz and Woronowicz, 1989; Jagannathan *et al.*, 1992). The latter is generated by $b_j$, $b_j^+$, $N_j$ ($j = 1, 2, \ldots, n$) and the corresponding commutation relations:

$$\begin{aligned} b_i^+ b_j^+ &= \sqrt{q} b_j^+ b_i^+, \qquad i < j \\ b_i b_j &= \frac{1}{\sqrt{q}} b_j b_i, \qquad i < j \\ b_i b_j^+ &= \sqrt{q} b_j^+ b_i, \qquad i \neq j \\ b_i b_i^+ &= 1 + q b_i^+ b_i + (1 - q) \sum_{k=i+1}^{n} b_k^+ b_k, \qquad i = 1, \ldots, n - 1 \end{aligned} \tag{9}$$

$$b_n b_n^+ = 1 + q b_n^+ b_n$$

$$[N_i, b_j^+] = +\delta_{ij} b_j^+, \qquad [N_i, b_j] = -\delta_{ij} b_j \qquad i, j = 1, \ldots, n$$

where $b_i (b_i^+)$ plays the role of annihilation (creation) operator and $N_i$ plays the role of number operator. Let us introduce the Fock-space basis $|n_1, n_2, \ldots, n_n\rangle$. The number operators $N_i$ $(i = 1, 2, \ldots, n)$ act as

$$N_i |n_1, n_2, \ldots, n_n\rangle = n_i |n_1, n_2, \ldots, n_n\rangle \tag{10}$$

and the operators $b_i$ $(b_i^+)$ act as

$$b_i |n_1, n_2, \ldots, n_n\rangle = \sqrt{q^{\Sigma_{k=i+1}^n n_k} [[n_i]]} |n_1, n_2, \ldots, n_i - 1, \ldots, n_n\rangle \tag{11}$$
$$b_i^+ |n_1, n_2, \ldots, n_n\rangle = \sqrt{q^{\Sigma_{k=i+1}^n n_k} [[n_i + 1]]} |n_1, n_2, \ldots, n_i + 1, \ldots, n_n\rangle$$

Then from the above relations we obtain the relation between the operators $b_i$ $(b_i^+)$ and the mode operators as follows:

$$b_i\, b_i^+ = q^{\Sigma_{k=i+1}^n N_k} [[N_i]], \qquad i \neq n \tag{12}$$

where the q-symbol $[[x]]$ is defined as

$$[[x]] = \frac{q^x - 1}{q - 1} \tag{13}$$

The state $|n_1, n_2, \ldots, n_n\rangle$ is obtained from the ground state $|0, 0, \ldots, 0\rangle$ by applying the creation operators as follows:

$$|n_1, n_2, \ldots, n_n\rangle = \frac{(b_n^+)^{n_n} \ldots (b_1^+)^{n_1}}{\sqrt{[[n_1]]!} \cdots \sqrt{[[n_n]]!}} |0, 0, \ldots, 0\rangle \tag{14}$$

Now we introduce the scale operators $Q_i = q^{N_i}$ $(i = 1, 2, \ldots, n)$ which act on the basis $|n_1\ n_2, \ldots, n_n\rangle$ as

$$Q_i |n_1, n_2, \ldots, n_n\rangle = q^{n_i} |n_1, n_2, \ldots, n_n\rangle \tag{15}$$

From the relations (9), we have

$$[b_i, (b_i^+)^m] = q^{1-m} [[m]]\ Q_i Q_{i+1} \cdots Q_n (b_i^+)^{m-1} \tag{16}$$

In the case where the parameter $q$ is chosen as the $K$th primitive root of unity, the dimension of the representation space becomes finite and we obtain the operator identities $(b_i^+)^K = 1$ and $(b_i)^K = 0$.

Now we turn to the $gl_q(n)$-covariant oscillator realization of the q-deformed centerless multi-Virasoro algebra. Let us introduce the operators

$$L_m^{(i)} = Q_i^{-1} \cdots Q_n^{-1} (b_i^+)^{m+1} b_i \tag{17}$$

For $m \geq -1$ and for negative values $m \leq -2$, we use the same formulas understood as monomials of $b_i^+$ with negative powers.

*Theorem.* The above generators $L_m^{(i)}$ for $m \in Z$, $i = 0, 1, \ldots, n-1$, satisfy the following q-commutation relations:

$$[L_r^{(i)}, L_m^i]_{(q^{(m-r)/2}, q^{(r-m)/2})} = [m - r]_{\sqrt{q}} L_{m+r}^{(i)}, \qquad i = 1, 2, \ldots, n-1$$
$$L_m^{(i)} L_{m'}^{(j)} = q^{mm'/2} L_{m'}^{(j)} L_m^{(i)}, \qquad i < j \tag{18}$$

where $[A, B]_{(q,p)} = qAB - pBA$.

These are exactly the relations of the q-deformed centerless multi-Virasoro algebra (8), where we have used the change $\sqrt{q} \rightarrow q$.

The proof is easily given by using (9) and (16) and algebraic manipulation.

*Remark.* Contrary to the undeformed case (5), the $n$-deformed copies are noncommuting. The property that these copies do not commute leads us to consider a new kind of symmetry deformation that is, to perform a deformation which also breaks the "chirality" of the theory.

In order to determine the central extension, we adopt the method used in Aizawa and Sato (1991). Let **K** be a commutative ring with a unit element and **A** be a **K**-module.

*Definition.* The set **A** is called deformation of the Lie algebra over **K** if there exists a bilinear mapping $(X, Y) \rightarrow [X, Y]$ $(X, Y \in \mathbf{A})$ such that

$$\mathbf{A} \times \mathbf{A} \rightarrow \mathbf{A}$$
$$(X, Y) \rightarrow [X, Y]$$

which satisfies the following conditions for all elements $X$, $Y$, and $Z \in \mathbf{A}$:

1. $[X, X] = 0$.
2. $[\epsilon(X), [Y, Z]] + [\epsilon(Y), [Z, X]] + [\epsilon(Z), [X, Y]] = 0$ (q-Jacobi identity).

Here $\epsilon$ is a deformation operator. We define the bracket product for $X = \Sigma_{n \in Z}\, \alpha_n^i\, L_n^{(i)}$ and $Y = \Sigma_{m \in z} \alpha_m'^j L_m^{(j)}$ $(i, j = 0, 1, \ldots, (n-1)$, where $\alpha_n^i$ and $\alpha_m'^j \in \mathbf{C}$, by

$$[X, Y] = (XY)_q = (YX)_q \tag{19}$$

and $\epsilon$ as

$$\epsilon(X) = \sum_{n \in Z} \alpha_n^i \frac{q^n + q^{-n}}{2} L_n^{(i)} \tag{20}$$

where

$$(L_m^{(i)} L_m^{(j)})_q = \frac{q^{mn} L_m^{(i)} L_n^{(j)} + q^{nm} L_n^{(j)} L_m^{(i)})}{2} \qquad \text{if} \quad i \neq j$$
$$= q^{n-m} L^{(j)}{}_n L_m^{(i)} \qquad \text{if} \quad i = j \tag{21}$$

One can easily prove that conditions 1 and 2 are satisfied, hence the q-algebra (18) is a deformation of multiconformal algebra over **C**.

To determine the central extension, we follow the method used in Aizawa and Sato (1991). Note that the central extension exists only in the case where $i = j$,

$$[L_m^{(i)}, L_n^{(i)}]_{(q^{(n-m)}, q^{(m-n)})} = [n - m]_q L_{m+n}^{(i)} + c_q^i(m, n) \qquad i, = 0, 1, \ldots, n - 1 \tag{22}$$

where $c_q^i(m, n)$ satisfies the following relation:

$$c_q^i(m, n) = -c_q^i(n, m) \tag{23}$$

From the q-deformed Jacobi identity, we obtain

$$(q^m + q^{-m})[n - l]c_q^i(m, n + l) + \text{cyclic permutation} = 0 \tag{24}$$

Following the method in Goddard and Olive (1986), and after algebraic manipulation, we find

$$c_q^i(m, n) = \frac{[m][m + 1][m]\,[m - 1]c^i}{[2m][3]!}\delta_{n+m,0} \tag{25}$$

where $c^i = (q^2 + q^{-2})\ c^i\ (2, -2)$. Then the classical limit of (25) is exactly the same as the usual center of the multiconformal algebra (6).

Next, we give the realization of the central extended q-multi-Virasoro algebra using the method of the operator product expansion of the energy-momentum tensor component $T(z^{(i)})$.

## 4. REALIZATION OF THE OPERATOR PRODUCT EXPANSION

In this section, we give the q-operators product expansion of the q-energy-momentum tensor component $T(z^{(l)})$, $l = 0, 1, \ldots, (n - 1)$. Let $L_m^{(l)}$ be the coefficient of the Laurent expansion of $T(z^{(l)})$

$$L_m^{(l)} = \frac{1}{2\pi i}\oint_0 dz^{(l)m+1}T(z^{(l)}) \tag{26}$$

where $T(z^{(l)})$ is the $l$th component of the multi-tensor energy-momentum, which is written as $T(z^{(l)}) = \Sigma_{(m\in Z)}\ L_m^{(l)} z^{(l)^{-m-2}}$.

Now, we introduce the q-product for two components of $T(Z(z^{(0)}, z^{(1)}, \ldots, z^{(n-1)})$:

$$(T(z^{(i)})T(w^{(j)}))_q = T(qz^{(i)})\ T(q^{-1}w^{(j)}) \qquad \text{for} \quad i = j \tag{27}$$

and for $i \neq j$, the q-product is defined such that

$$(L_m^{(i)} L_n^{(j)})_q = \frac{(q^{mn} L_m^{(i)} L_n^{(j)} + q^{nm} L_n^{(j)} L_m^{(m)}}{2} \tag{28}$$

Now let us limit ourselves to the bracket of $L_m^{(i)}$ and $L_n^{(j)}$ for $i = j$, which is expressed as

$$\begin{aligned}
[L_m^{(i)}, L_n^{(j)}] &= \frac{1}{(2\pi i)^2} \oint_0 dz^{(i)} \oint_0 dw^{(i)} \\
&\quad \times z^{(i)m+1} w^{(i)n+1} (T(z^{(i)}) T(w^{(i)}))_q - (T(w^{(i)}) T(z^{(i)}))_q \\
&= \frac{1}{(2\pi i)^2} \oint_0 dz^{(i)} \oint_P dw^{(i)}\, z^{(i)m+1} w^{(i)n+1} (T(z^{(i)}) T(w^{(i)}))_q
\end{aligned} \tag{29}$$

where $P$ means the poles of the integrand $(T(z^{(i)})T(w^{(i)}))_q$. We assume at that the pole structure of $(T(z^{(i)})T(w^{(i)}))_q$ is

$$\begin{aligned}
(T(z^{(i)}) T(w^{(i)}))_q &= \frac{[w^{(i)} \partial_{w^{(i)}}^{(i)}]}{w^{(i)2} [2 w^{(i)} \partial_{w^{(i)}}^{(i)}]} w^{(i)2} \frac{c^i}{(z^{(i)} - w^{(i)})_q^4} \\
&\quad + \frac{1}{(z^{(i)} - w^{(i)})} \left( \frac{T(q^{-1} w^{(i)})}{q z^{(i)} - q^{-1} w^{(i)}} + \frac{T(q w^{(i)})}{q^{-1} z^{(i)} - q w^{(i)}} \right) \\
&\quad + \frac{1}{z^{(i)} - w^{(i)}} (\partial_{w^{(i)}}^{(i)})_q T(w^{(i)})
\end{aligned} \tag{30}$$

where $(z^{(i)} - w^{(i)})_q^n$ and $\partial_q^{(i)}$ are the q-analogues of the distance and the $i$th derivative, such that

$$\begin{aligned}
(z^{(i)} - w^{(i)})_q^n &= \prod_{k=1}^{n} (z^{(i)} - w^{(i) q^{n-2k+1}}) \\
\partial_q^{(i)} &= \frac{T(q z^{(i)}) - T(q^{-1} z^{(i)})}{(q - q^{-1}) z^{(i)}}
\end{aligned} \tag{31}$$

Inserting (30) into (29) gives

$$[L_m^{(i)}, L_n^{(i)}] = A_1 + A_2 + A_3 \tag{32}$$

where $A_{1,2,3}$ are given by

$$\begin{aligned}
A_1 &= \frac{1}{(2\pi i)^2} \oint_0 dz^{(i)} \oint_P dw^{(i)}\, z^{(i)m+1} w^{(i)n+1} \\
&\quad \times (q^{n+1} + q^{-n-1}) \frac{1}{(z^{(i)} - w^{(i)})_n^2} T(w^{(i)})
\end{aligned}$$

$$A_2 = \frac{1}{(2\pi i)^2} \oint_0 dz^{(i)} \oint_P dw^{(i)}\, z^{(i)m+1} w^{(i)n+1} \frac{1}{(z^{(i)} - w^{(i)})}\, \partial_q^{(i)}\, T(w^{(i)})$$

$$A_3 = \frac{1}{(2\pi i)^2} \oint_0 dz^{(i)} \oint_P dw^{(i)}\, z^{(i)m+1} w^{(i)n+1} \frac{[m]}{[2m]} \frac{1}{(z^{(i)} - w^{(i)})_q^4}$$

In order to compute these integrals it is convenient to use

$$\oint_0 dw^{(i)}\, w^{(i)n-1}\, F(w^{(i)}\, \partial_q^{(i)}) G(w^{(i)}) = \oint_0 dw^{(i)}\, w^{(i)n-1} F(-n) G(w^{(i)})$$

and the q-deformed residue formula

$$\frac{1}{[n]!} \partial_q^{(i)n}\, f(z^{(i)}) = \frac{1}{(2\pi i)} \oint_P dw^{(i)} \frac{f(z^{(i)})}{(w^{(i)} - z^{(i)})_q^{n+1}}$$

After algebraic manipulation we reach the same equations as (22) and (25).

Therefore, we conclude that the q-operator product expansion given in (30) is a realization of a q-deformed multiconformal algebra with a central charge.

## 5. CONCLUSION AND OUTLOOK

In this paper, I presented a q-deformation of multiconformal algebra for the centerless case. The $gl_q(n)$-covariant oscillators were realized for this case. By introducing the q-deformation of the Jacobi identity, we obtain the q-analogue of central charge. The realization of the q-deformation of multiconformal algebra with charge was given in terms of the q-operator product expansion. Finally, I hope that the introduction of the q-operators product expansion, which is a powerful tool in multiconformal field theory, will be useful for q-deformed multiconformal field theory and for the construction of q-deformed extended objects.

## REFERENCES


Aizawa, A., and Sato, H. (1991). *Physics Letters B* **256**, 185.

Alvarez-Gaume´, L., Gomes, C., and Sierra, G. (1998). *Physics Letters B* **220**, 142.

Bernard, D., LeCaire, A. (1991). In *Communications in Mathematical Physics* **142**, 99.

Biedenharn, L. C. (1989). *Journal of Physics A* **22**, 873.

Chaichian, M., and Kulish, P. (1990). *Physics Letters B* **234**, 72.

Chaichian, M., Ellinas, D., and Kulish, P. (1990). *Physical Review Letters* **65**, 980.

De Vega, H. (1989). *International Journal of Modern Physics A* **4**, 2371.

Drinfel'd, V. (1986). In *Proceedings International Congress of Mathematicians, Berkeley*, p. 78.

Faddeev, L. D., Reshetikhin, N. Yu., and Takhtajan, L. (1987). In *Braid Groups, Knot Theory and Statistical Mechanics*, C. N. Yang and M. L Ge, eds. World Scientific, Singapore.
Fleury, F., Rausch de Traubenberg, M., and Yamaleev, R. M. (1993). *Journal of Mathematical Analysis and Application* **180**, 431–457.
Fleury, F., Rausche de Traubenberg, M., and Yamaleev, R. M. (1995). *Journal of Mathematical Analysis and Application* **191**, 118–136.
Goddard, P., and Olive, D. (1986), *International Journal of Modern Physics A* **1**, 303.
Greenberg, O. W. (1990). *Physical Review Letters* **64**, 705.
Jagannathan, R., *et al.* (1992). *Journal of Physics A* **25**, 6429.
Jimbo, M. (1985). *Letters in Mathematical Physics* **10**, 63.
Jimbo, M. (1986). *Letters in Mathematical Physics* **11**, 247.
Kulish, P. P., and Damaskinsky, E. (1990). *Journal of Physics A* **23**, 415.
Macfarlane, A. J. (1989). *Journal of Physics A* **22**, 4581.
Manin, Yu. I. (1988). Quantum groups and non-commutati ve geometry, Centre de Recherches Mathématiques Montréal.
Manin, Yu. I. (1991). *Topics in Non-Commutative Geometry*, Princeton University Press, Princeton, New Jersey.
Mohapata, R. N. (1990). *Physics Letters B* **242**, 407.
Moore, G., and Reshetkin, N. Yu. (1989). *Nuclear Physics B* **220**, 557.
Ouarab, A., El Kinani, E. H., and Zakkari, M. (1998). *International Journal of Theoretical Physics* **37**, 1241.
Pasquier, V., and Saleurs, H. (1990). *Nuclear Physics B* **330**, 523.
Pusz, W., and Woronowicz, S. L. (1989). *Reports on Mathematical Physics* **27**, 231.
Wess, J., and Zumino, B. (1990). *Nuclear Physics B* **18**, 302.
Woronowicz, S. L. (1987). *Communications in Mathematical Physics* **111**, 613.